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On the master symmetries and bi-Hamiltonian structure of the Toda lattice

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Abstract. Starting from a conformal symmetry, higher-order Poisson tensors, deformation relations and master symmetries for the Toda lattice are obtained. A hierarchy of time-dependent symmetries is also constructed. Using reduction, deformation relations previously known to hold up to a certain equivalence relation are shown to be exact.

Introduction

In Damianou (1990), master symmetries and deformation relations for the Toda lattice were constructed, and its connection with the R-matrix approach given. On the other hand, a well known theorem of Oevel (1987) (see also theorem 2.1) relates master symmetries to a conformal symmetry of the system, when a recursion operator is available. In this paper we relate the two approaches using a recursion operator and reduction. In particular, deformation relations previously known to hold up to a certain equivalence relation are shown to be exact.

Our approach consists in working in physical variables and then reducing to Flaschka's variables. Although the recursion operator itself cannot be reduced (this is also observed in Morosi and Tandi (1990)), the deformation relations and master symmetries do reduce. One advantage of this approach is that it immediately yields a hierarchy of time-dependent symmetries of the Toda lattice. These symmetries also appear in Damianou (1993).

1. Master symmetries for differential equations

We recall some basic facts of the theory of master symmetries for differential equations. More details can be found in Fuchssteiner (1983). Consider a differential equation on a manifold M:

$$\dot{x} = X(x). \tag{1.1}$$

As usual, a vector field Y is a symmetry of (1.1) if

$$[Y, X] = 0. \tag{1.2}$$

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More generally, a family $Y = Y(x, t)$ of vector fields depending smoothly on t is a time-dependent symmetry of (1.1) if

$$\frac{\partial Y}{\partial t} + [Y, X] = 0. \tag{1.3}$$

We should view Y as a time-dependent vector field.

We can generalize (1.2) as follows. A vector field Z is called a *generator of degree n* if

$$[[\ldots[Z, X], \ldots], X], X]] = 0.$$

If Z is a generator of degree n then the time-dependent vector field

$$Y_Z = \exp(\text{ad } X)Z = \sum_{k=0}^{n} \frac{t^k}{k!}[[\ldots[Z, X], \ldots], X], X]] \tag{1.4}$$

satisfies (1.3), and so is a time-dependent symmetry of (1.1). Thus, t time-dependent symmetries which are polynomial in t are in 1:1 correspondence with generators of degree n.

Generators satisfy the following properties:

(i) if Z is a generator of degree n, then $[Z, X]$ is a generator of degree $n - 1$;

(ii) if Z_1 and Z_2 are generators of degree n_1 and n_2, then $[Z_1, Z_2]$ is a generator of degree $n_1 + n_2 - 1$;

(iii) a symmetry is a generator of degree 0.

In particular we see that the set of all generators form a Lie subalgebra of the algebra of all vector fields $\mathcal{X}(M)$. We shall call a generator of degree 1 a *master symmetry*. Thus the condition for Z to be a master symmetry is

$$[[Z, X], X] = 0 \qquad \text{and} \qquad [Z, X] \neq 0.$$

Proposition 1.1. Let Z be a master symmetry. Then

(i) $[Z, X]$ is an ordinary symmetry;

(ii) $[Z, [Z, X]]$ is an ordinary symmetry;

Proof. It is obvious from the definitions and the Jacobi identity. □

In general, given a master symmetry all we get is the two symmetries given in the proposition. Under an additional assumption we can generate further symmetries as follows.

Proposition 1.2. Suppose Y is a symmetry of (1.1) which commutes with every other symmetry, and let Z be a master symmetry. Then $[Z, Y]$ is also a symmetry.

Proof. Use the Jacobi identity again. □

2. Master symmetries and bi-Hamiltonian systems

On a manifold M on which the first cohomology vanishes, we consider a bi-Hamiltonian vector field

$$X_1 = J_1 \mathrm{d}H_0 = J_0 \mathrm{d}H_1 \tag{2.1}$$

where J_i are compatible Poisson tensors, and H_i are the Hamiltonian functions. We assume that J_0 is symplectic, so we can define the recursion operator

$$N = J_1 J_0^{-1}. \tag{2.2}$$

Recall that N is a tensor of type (1,1) with vanishing Nijenhuis torsion. There is a whole hierarchy of higher-order flows associated with the vector fields

$$X_i = N^{i-1} X_1 \qquad i = 1, 2, \ldots. \tag{2.3}$$

If we introduce the higher-order Poisson tensors

$$J_i = N^i J_0 \qquad i = 1, 2, \ldots \tag{2.4}$$

and define the Hamiltonians $\{H_i\}$ by

$$\mathrm{d}H_i = (N^*)^i \mathrm{d}H_0 \qquad i = 1, 2, \ldots$$

where N^* denotes the adjoint of N, then the higher-order flows have the multi-Hamiltonian formulation

$$X_{i+j} = J_i \mathrm{d}H_j. \tag{2.5}$$

Magri's (1978) theorem states that the flows (2.3) all commute with each other, and the functions $\{H_i\}$ form a sequence of first integrals of each flow, in involution with respect to both Poisson brackets.

For a bi-Hamiltonian system master symmetries can be obtained from the following result due to Oevel (1987):

Theorem 2.1. Suppose that Z_0 is a conformal symmetry for both J_0, J_1 and H_0, i.e. for some scalars α, β, and γ we have

$$L_{Z_0} J_0 = \alpha J_0 \qquad L_{Z_0} J_1 = \beta J_1 \qquad L_{Z_0} H_0 = \gamma H_0.$$

Then the vector fields

$$Z_i = N^i Z_0$$

satisfy

(i) $[Z_i, X_j] = (\beta + \gamma + (j-1)(\beta - \alpha)) X_{i+j}$

(ii) $[Z_i, Z_j] = (\beta - \alpha)(j - i) Z_{i+j}$

(iii) $L_{Z_i} J_j = (\beta + (j - i - 1)(\beta - \alpha)) J_{i+j}$.

The set of first integrals $\{H_i\}$ can be obtained from the formula

(iv) $(\mathrm{d}H_j, Z_i) = (\gamma + (i + j)(\beta - \alpha)) H_{i+j}$.

A proof of this result can be found in Oevel (1987). Using the methods of section 1 we immediately obtain

Corollary 2.2. Under the hypothesis of the theorem, for each integer $i = 1, 2, \ldots,$ the vector fields

$$Y_{Z_j} = Z_j + t(\beta + \gamma + (i-1)(\beta - \alpha))X_{i+j} \qquad j = 1, 2, \ldots$$

are time-dependent symmetries of the i*th-order flow.*

Proof. Each $Z_j, j = 1, 2, \ldots,$ is a master symmetry. Using (1.4) and relation (i) of theorem 2.1 we compute

$$Y_{Z_j} = Z_j + t[Z_j, X_i] = Z_j + t(\beta + \gamma + (i-1)(\beta - \alpha))X_{i+j}. \qquad \square$$

3. The Toda lattice

We consider the finite, non-periodic, Toda lattice, i.e. a system of particles on the line under exponential interaction with nearby particles. It has the following bi-Hamiltonian formulation:

$$\begin{aligned}
J_0 &= \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \\
J_1 &= \sum_{i=1}^{n-1} 2\mathrm{e}^{2(q^i - q^{i+1})} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i<j} \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial q^i} \qquad (3.1) \\
H_0 &= \sum_{i=1}^{n} p_i \qquad H_1 = \sum_{i=1}^{n} \frac{{p_i}^2}{2} + \sum_{i=1}^{n-1} \mathrm{e}^{2(q^i - q^{i+1})}.
\end{aligned}$$

Note that J_0 is symplectic. The recursion operator is then

$$\begin{aligned}
N = &\sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \sum_{i=1}^{n-1} 2\mathrm{e}^{2(q^i - q^{i+1})} \left(\frac{\partial}{\partial p_{i+1}} \otimes \mathrm{d}q^i - \frac{\partial}{\partial p_i} \otimes \mathrm{d}q^{i+1} \right) \\
&+ \frac{1}{2} \sum_{i<j} \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}p_j - \frac{\partial}{\partial q^j} \otimes \mathrm{d}p_i \right) + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i. \qquad (3.2)
\end{aligned}$$

We will now show that the vector field

$$Z_0 = \sum_{i=1}^{n} \frac{n+1-2i}{2} \frac{\partial}{\partial q^i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \qquad (3.3)$$

is a conformal symmetry for both J_0, J_1 and H_0, so we will be able to apply theorem 2.1 and its corollary.

In fact, we compute

$$\begin{aligned}
L_{Z_0} J_0 &= \sum_{i,j=1}^{n} \left[\frac{n+1-2j}{2} \frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \right] + \sum_{i,j=1}^{n} \left[p_j \frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \right] \\
&= 0 - \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} = -J_0.
\end{aligned}$$

Next observe that Z_0 is a Hamiltonian vector field with respect to J_1 for the Hamiltonian $I = \sum_i q^i$, so we have

$$L_{Z_0} J_1 = 0.$$

Finally, a simple computation shows that $L_{Z_0} H_0 = H_0$. Therefore theorem 2.1 holds with $\alpha = -1, \beta = 0, \gamma = 1$. It follows that the higher-order Poisson tensors for the Toda lattice satisfy the deformation relations:

$$L_{Z_i} J_j = (j - i - 1) J_{i+j} \tag{3.4}$$

$$L_{Z_i} H_j = (i + j + 1) H_{i+j} \tag{3.5}$$

where $Z_i \equiv N^i Z_0$ satisfy

$$[Z_i, Z_j] = (j - i) Z_{i+j}. \tag{3.6}$$

If we denote by X_i the Hamiltonian vector field generated by H_i, with respect to J_0, we also have

$$[Z_i, X_j] = j X_{i+j} \tag{3.7}$$

and from the corollary we obtain the time-dependent symmetries

$$Y_{Z_j} \equiv Z_j + it X_{i+j} \qquad j = 1, 2, \ldots. \tag{3.8}$$

Another multi-Hamiltonian formulation is known for Toda lattice in terms of the Flaschka's variables. Recall that the Flaschka transformation is the map $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ defined by

$$(q^1, \ldots, q^n, p_1, \ldots, p_n) \mapsto (a_1, \ldots, a_{n-1}, b_1, \ldots, b_n)$$

where $a_i = \mathrm{e}^{(q^i - q^{i+1})}, b_i = p_i$. The Poisson tensors J_0 and J_1 reduce to R^{2n-1}. This can be checked directly by setting

$$\tilde{J}_0 = \sum_{i=1}^{n-1} a_i \left(\frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_i} - \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_{i+1}} \right)$$

$$\begin{aligned} \tilde{J}_1 = \sum_{i=1}^{n-1} \left(a_i b_i \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_i} - a_i b_{i+1} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_{i+1}} \right) \\ + \sum_{i=1}^{n-1} 2a_i{}^2 \frac{\partial}{\partial b_{i+1}} \wedge \frac{\partial}{\partial b_i} + \sum_{i=1}^{n-2} \frac{1}{2} a_i a_{i+1} \frac{\partial}{\partial a_{i+1}} \wedge \frac{\partial}{\partial a_i} \end{aligned} \tag{3.9}$$

and observing that the projection $\pi : (\mathbb{R}^{2n}, J_i) \to (\mathbb{R}^{2n-1}, \tilde{J}_i)$ is a Poisson morphism.

The bi-Hamiltonian formulation for the Toda lattice in Flaschka's variables is exactly the one defined by the Poisson tensors (3.9) and the reduced Hamiltonians

$$\tilde{H}_0 = \sum_{i=1}^{n} b_i \qquad \tilde{H}_1 = \sum_{i=1}^{n} \frac{b_i{}^2}{2} + \sum_{i=1}^{n-1} a_i{}^2.$$

There is, however, a big difference between the original bi-Hamiltonian formulation and the reduced bi-Hamiltonian formulation. The Poisson structures $\tilde{J}_0$ and $\tilde{J}_1$ are not symplectic, and so there is no obvious recursion operator. In fact, the recursion operator N given by (3.2) cannot be reduced. This is most easily seen using the notion of projectable vector field. Recall that a vector field Z is *projectable* if for every vector field Y tangent to the fibres $\pi^{-1}(x)$, the vector field $L_Y Z$ is also tangent to the fibres. If that is the case, the vector field Z can be reduced to a vector field $\tilde{Z}$ given by $\tilde{Z}(\pi(x)) = \mathrm{d}\pi(x) \cdot Z(x)$. Conversely, any vector field on the reduced space is the image by π of a projectable vector field.

Now we claim that N does not map projectable vector fields to projectable vector fields, as is necessary for the reduction to work. To prove this we note that the fibres are the lines in $\mathbb{R}^{2n}$ parallel to the vector $(1, \ldots, 1, 0, \ldots, 0)$, so a vector field Z is projectable if for every function $f \in C^\infty(M)$ there exists a function $g \in C^\infty(M)$ such that

$$\left[Z, f \sum_i \frac{\partial}{\partial q^i}\right] = g \sum_i \frac{\partial}{\partial q^i}.$$

For example, the vector field $\partial/\partial q^i$ is projectable but

$$N \frac{\partial}{\partial q^i} = p_i \frac{\partial}{\partial q^i} - 2\mathrm{e}^{2(q^{i-1}-q^i)} \frac{\partial}{\partial p^{i-1}} + 2\mathrm{e}^{2(q^i-q^{i+1})} \frac{\partial}{\partial p^{i+1}}$$

is not projectable. We conclude that no recursion operator exists relating the two reduced Poisson tensors.

In spite of the fact that there is no recursion operator for the reduced Toda lattice, higher-order Poisson structures are known, and they satisfy certain deformation relations [1]. This can be explained by the following result.

Theorem 3.1. The vector fields $Z_i = N^i Z_0, i = 0, 1, 2, \ldots,$ are projectable. The corresponding reduced vector fields satisfy

$$[\tilde{Z}_i, \tilde{Z}_j] = (j - i)\tilde{Z}_{i+j}. \tag{3.10}$$

In particular, the higher-order Poisson tensors can be reduced to Poisson tensors $\tilde{J}_i$, satisfying the deformation relations

$$L_{Z_i} \tilde{J}_j = (j - i - 1)\tilde{J}_{i+j}. \tag{3.11}$$

There are also reduced Hamiltonians $\{\tilde{H}_i\}$ and reduced higher-order flows $\tilde{X}_i$ satisfying

$$L_{\tilde{Z}_i} \tilde{H}_j = (i + j + 1)\tilde{H}_{i+j} \tag{3.12}$$

$$[\tilde{Z}_i, \tilde{X}_j] = j\tilde{X}_{i+j}. \tag{3.13}$$

Proof. All we have to prove is that the vector fields $\tilde{Z}_i$ are projectable, so that all the hierarchy can be reduced. The rest of the proposition follows from relations (3.4)–(3.7).

We compute

$$\left[N, \sum_{i=1}^n \frac{\partial}{\partial q^i}\right] = \sum_{i=1}^{n-1} 2\mathrm{e}^{2(q^i-q^{i+1})} \left(\frac{\partial}{\partial p_i} \otimes \mathrm{d}q^{i+1} - \frac{\partial}{\partial p_{i+1}} \otimes \mathrm{d}q^i\right)$$
$$- \sum_{i=2}^{n} 2\mathrm{e}^{2(q^{i-1}-q^i)} \left(\frac{\partial}{\partial p_{i-1}} \otimes \mathrm{d}q^i - \frac{\partial}{\partial p_i} \otimes \mathrm{d}q^{i-1}\right) = 0.$$

Therefore, for any $f \in C^\infty(M)$, we find

$$\left[N^i Z_0, f\sum_i \frac{\partial}{\partial q^i}\right] = (N^i Z_0)(f)\sum_i \frac{\partial}{\partial q^i} - f\left[N^i Z_0, \sum_i \frac{\partial}{\partial q^i}\right]$$
$$= g\sum_i \frac{\partial}{\partial q^i} - fN^i\left[Z_0, \sum_i \frac{\partial}{\partial q^i}\right] = g\sum_i \frac{\partial}{\partial q^i}$$

so the $Z_i = N^i Z_0,\ i = 0, 1, 2, \ldots$, are projectable. □

The deformation relations (3.11) were known to hold up to a certain equivalence relation Damianou (1990). Our proof shows that they are, in fact, exact. We note that the master symmetries $\{\tilde{Z}_i\}$, for $i \geqslant 2$, are different from the master symmetries given in Damianou (1990). However, for $i = 1$ they differ by a multiple of the Hamiltonian vector field X_1, and so the higher order reduced Poisson tensors (3.11), coincide with the ones given in Dadmianou (1990).

It follows, exactly as in corollary 2.2, that we have a hierarchy of reduced time-dependent symmetries:

Corollary 3.2. For each integer $i = 1, 2, \ldots$, the vector fields

$$Y_{\tilde{Z}_j} \equiv \tilde{Z}_j + it\tilde{X}_{i+j} \qquad j = 1, 2 \ldots$$

are time-dependent symmetries of the ith-order Toda flow.

We have learned during the preparation of this manuscript that corollary 3.2 had also been obtained in Damianou (1993), although by different methods.

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